

# Turchin's Relation and Subsequence Relation on Traces Generated by Prefix Grammars<sup>\*</sup>

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**Abstract.** Turchin's relation was introduced in 1988 by V. F. Turchin for loop approximation in supercompilation. The paper studies properties of an analogue of the Turchin relation and properties of the subsequence embedding on a restricted set of traces generated by prefix grammars or by a product of prefix grammars.

**Keywords:** Higman's lemma, prefix rewriting, well binary relation, computational complexity, termination, supercompilation

## 1 Introduction

In computer science the homeomorphic embedding is investigated from two completely different points of view, for it is of both theoretical and practical interest.

On the one hand, the embedding showed itself to be useful as a branch termination criterion in constructing tools for program transformation ([14], [3]). What makes the homeomorphic embedding reasonable as a termination criterion is the non-existence of an infinite sequence of finite labeled trees such that no tree in the sequence is embedded into some its derivative (the fact was proved by Kruskal and is called Kruskal's theorem; for an elegant proof of the fact see [8]). Relations with this property are called well-binary relations.

**Definition 1.**  $R, R \subset S \times S$ , is called **a well binary relation**, if every sequence  $\{\Phi_n\}$  of elements from  $S$  such that  $\forall i, j (i < j \Rightarrow (\Phi_i, \Phi_j) \notin R)$  is finite. If  $R$  is well binary and transitive it is called a well quasiorder (wqo).

A sequence  $\{\Phi_n\}$  with the property  $\forall i, j (i < j \Rightarrow (\Phi_i, \Phi_j) \notin R)$  is called **a bad sequence** with respect to  $R$ . Thus, well-binariness of  $R$  can be formulated equivalently as "all bad sequences with respect to  $R$  are finite".

On the other hand, well-binariness of the homeomorphic embedding is shown to be non-provable in the Peano arithmetic with the first-order induction scheme [13], and this fact aroused interest of logicians and computer scientists with background in mathematical logic (a thorough study of the proof-theoretical strength of the fact is in [17]). Studies of the homeomorphic embedding as a termination

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criterion for term rewriting systems ([12, 15]) are located in the middle between these poles of pure theory and practice.

The problem is that since these two domains live their own separate lives, it is not always obvious how to use the theoretical investigations in the practical program transformations. Theoricians study properties of the homeomorphic embedding (and similar relations) on arbitrary (maybe not even computable) sequences of trees, and that can imply somewhat obscure view on practical features of the relations: in particular it was established that the upper bound on a bad sequence length with respect to the homeomorphic embedding dominates every multiple recursive function [13], which looks redundantly from the practical point of view. But in real applications the opposite problem becomes much more frequent: the homeomorphic embedding yields branch termination too early [7, 12]. In some algorithms of program analysis this flaw was partially fixed either by making an additional annotation [6] or by intersecting the embedding with other wqos [1].

In this paper we study properties of a special case of the homeomorphic embedding on a restricted set of computable sequences.

**Definition 2.** *Having two words  $\Phi, \Psi$  in an alphabet  $\Upsilon$  let us say that  $\Phi$  is embedded in  $\Psi$  **with respect to the subsequence relation** ( $\Phi \trianglelefteq \Psi$ ) ( $\trianglelefteq$  is also called the scattered subword relation) if  $\Phi$  is a subsequence of  $\Psi$ .*

The subsequence relation is proved to be a well quasiorder by G. Higman [4]. We prove that while applied only to sequences generated by prefix grammars the relation admits bad sequences not more than exponential over a grammar size. If we apply the relation to a direct product of sequences generated by prefix grammars we receive the multiple recursive upper bound found by H. Touzet [15]. Also we show how to make a refinement of the subsequence relation that solves the empty word problem for languages generated by alphabetic prefix grammars and inherits some useful features of Turchin's relation, which was also used in program transformation (in particular, in the supercompiler SCP4 [9]).

The paper is organized as follows. First, we introduce notion of a prefix grammar. Then we give a definition of the Turchin relation and shortly prove its well-binariness. After that we show how to build maximal bad sequences with respect to the Turchin relation and give some discussion on using this relation combined with other well binary relations. Finally, we show how our refinement for the Turchin relation allows to refine the subsequence relation and, using our knowledge about the Turchin relation, we investigate properties of the subsequence relation on traces generated by prefix grammars.

The main contributions of the paper are the following:

1. We outline the concept of Turchin's relation in terms of prefix grammars and investigate properties of the relation.
2. We link Turchin's relation with the subsequence relation and show how to model the former by the latter not using a notion of time for sequences generated by prefix grammars.

3. We determine upper bounds of bad sequence length with respect to both relations for sequences generated by a single prefix grammar and for direct products of two sequences generated by prefix grammars.
4. We show that a minimal natural well binary generalization of Turchin's relation on direct products of sequences generated by prefix grammar is the subsequence relation.

## 2 Prefix Grammars

We consider a restricted class of generative indeterministic grammars, in which rewriting rules are applied in an arbitrary order.

**Definition 3.** A tuple  $\langle \mathcal{Y}, \mathbf{R}, \Gamma_0 \rangle$ , where  $\mathcal{Y}$  is an alphabet,  $\Gamma_0 \in \mathcal{Y}^+$  is an initial word, and  $\mathbf{R} \subset \mathcal{Y}^+ \times \mathcal{Y}^*$  is a finite set of rewrite rules<sup>1</sup>, is called a **prefix grammar** if  $R : R_l \rightarrow R_r$  can be applied only to words of the form  $R_l\Phi$  (where  $R_l$  is a prefix and  $\Phi$  is a (possibly empty) suffix) and generates only words of the form  $R_r\Phi$ .

If the left-hand side  $R_l$  of a rule  $R : R_l \rightarrow R_r$  has the length 1 (only the first letter is rewritten) then the prefix grammar is called an **alphabetic prefix grammar**.

A **trace** of a prefix grammar  $\mathbf{G} = \langle \mathcal{Y}, \mathbf{R}, \Gamma_0 \rangle$  is a word sequence  $\{\Phi_i\}$  (finite or infinite) where  $\Phi_1 = \Gamma_0$  and for all  $i \exists R(R : R_l \rightarrow R_r \ \& \ R \in \mathbf{R} \ \& \ \Phi_i = R_l\Theta \ \& \ \Phi_{i+1} = R_r\Theta)$  ( $\Theta$  is a suffix). In other words, the elements of a trace are derived from their predecessors by applications of rewrite rules from  $\mathbf{G}$ .

**Example 1** Consider the following prefix grammar  $\mathbf{G}_\Lambda$  with  $\mathcal{Y} = \{a, b, c\}$  and the following rewrite rules:

$$\begin{aligned}
 R^{[1]} : \Lambda &\rightarrow ba & R^{[2]} : b &\rightarrow \Lambda & R^{[3]} : aac &\rightarrow \Lambda \\
 R^{[4]} : aad &\rightarrow \Lambda
 \end{aligned}$$

We cannot apply the rule  $R^{[3]}$  to **baacb**, for **baacb** starts not by **aac**. If we apply  $R^{[1]}$  or  $R^{[2]}$  to **baacb** the only correct results of the applications are **babaacb** and **aacb** respectively.

When V. F. Turchin discussed a search of semantic loops in Refal programs he considered a stack model, which resembles a prefix grammar [16]. V. F. Turchin proposed to observe call stack configurations to prevent infinite unfolding of a special sort. He aimed at cutting off branches where a stack top derives a path that ends with the same stack top. If we denote the stack top as  $\Phi$ , the derivation of  $\Phi$  with  $\Phi$  on the top as  $\Phi\Psi$ , and the part of the initial stack that is not modified as  $\Theta$  then we can say that a branch is dangerous with respect to Turchin's relation if it contains pairs of the form  $\Phi\Theta, \Phi\Psi\Theta$ . We can notice that the terms form a pair with respect to the subsequence relation, but V. F. Turchin proposed a stronger

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<sup>1</sup> It is usually said that  $\mathcal{Y}$  is finite, but this restriction is unnecessary in our case. Only finiteness of  $\mathbf{R}$  matters in our study.

relation for more precise identification of such stack configurations in his work [16]. V.F. Turchin used this relation to construct better loop approximations in residual programs, but the relation can be also used to forbid a program transformation process to halt driving on finite computation branches. The last property is analyzed in this paper for prefix-grammar-generated traces.

### 3 Turchin’s Relation

To describe the Turchin relation for grammar-generated traces we use a formalization presented in [9]. The formalization introduces a notion of time indices. The main idea of the formalization is to mark every letter in the trace by a natural number that points to the position in the trace where the letter first appears. The order of words in a trace is from up to down.

The length of  $\Phi$  is denoted by  $|\Phi|$ .

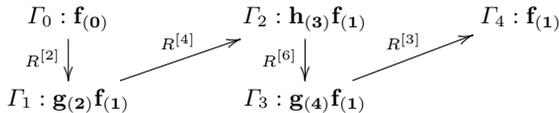
**Definition 4.** Consider a trace  $\{\Phi_i\}$  generated by a prefix grammar  $\mathbf{G}$ ,  $\mathbf{G} = \langle \Upsilon, \mathbf{R}, \Gamma_0 \rangle$ . Supply letters of  $\Phi_i$  by numbers that correspond to their **time indices** as follows. The  $i$ -th letter of  $\Gamma_0$  is marked by the number  $|\Gamma_0| - i$ ; if the maximal time index in the trace  $\{\Phi_i\}_{i=1}^k$  is  $M$  and  $\Phi_{k+1}$  is derived from  $\Phi_k$  by an application of  $R : R_l \rightarrow R_r$  then the  $i$ -th letter  $\Phi_{k+1}$  ( $i \leq |R_r|$ ) is marked by  $M + |R_r| - i + 1$ . Time indices of the other letters of  $\Phi_{k+1}$  coincide with the corresponding time indices of  $\Phi_k$ .

We call such annotation time indexing and we call a trace with the annotation **a computation**.

**Example 2** Let us consider a grammar  $\mathbf{G}_{\text{LOG}}$  with  $\Upsilon = \{f, g, h\}$  and the following rewrite rules:

$$\begin{array}{lll} R^{[1]} : f \rightarrow \Lambda & R^{[3]} : g \rightarrow \Lambda & R^{[5]} : h \rightarrow \Lambda \\ R^{[2]} : f \rightarrow gf & R^{[4]} : g \rightarrow h & R^{[6]} : h \rightarrow g \end{array}$$

$\Gamma_0 = f$ . A first segment of a computation yielded by the grammar  $\mathbf{G}_{\text{LOG}}$  can look as:



The time indices are in subscripts, enclosed in brackets. Note that the letter  $f_{(0)}$  in  $\Gamma_0$  is replaced by  $f_{(1)}$  in  $\Gamma_1$ , and  $f_{(0)} \neq f_{(1)}$ .

In the sequel Greek capitals ( $\Gamma, \Delta, \Theta, \Psi, \Phi$ ) denote words in a computation (with the time indexing).  $\Delta[k]$  denotes the  $k$ -th letter of  $\Delta$  (counting from the beginning).

An equivalence up to the time indices is formally defined as follows.  $\Phi \approx \Psi$  if  $|\Phi| = |\Psi|$  and  $\forall i (i \geq 1 \ \& \ i \leq |\Phi| \Rightarrow (\Phi[i] = a_{(n)} \ \& \ \Psi[i] = b_{(m)} \Rightarrow a = b))$ . The definition has the following simple meaning: if we erase time indices of all letters in  $\Phi$  and  $\Psi$  then  $\Phi$  and  $\Psi$  will coincide literally. For instance, in Example 2  $f_{(0)} \neq f_{(1)}$ , but  $f_{(0)} \approx f_{(1)}$ .

Now we are ready to define **Turchin's relation**  $\Gamma \preceq \Delta$ . Loosely speaking, it includes pairs  $\langle \Gamma, \Delta \rangle$ , where  $\Gamma$  can be presented as [Top] [Context],  $\Delta$  can be presented as [Top] [Middle] [Context], and the suffix [Context] is not modified in the computation segment thar starts from  $\Gamma$  and ends with  $\Delta$ .

**Definition 5.**  $\Gamma \preceq \Delta \Leftrightarrow \Gamma = \Phi\Theta_0 \ \& \ \Delta = \Phi'\Psi\Theta_0 \ \& \ \Phi' \approx \Phi$ . Pairs  $\Gamma, \Delta$  such that  $\Gamma \preceq \Delta$  are called **Turchin pairs**<sup>2</sup>.

$\preceq$  is not transitive but it is reflexive and antisymmetric up to  $\approx$  [11]. Well-binarity of the relation can be proved using the following observation. If a rule  $R$  has a non-empty right-hand side  $R_r$ ,  $\Phi \approx R_r$ ,  $\Phi' \approx R_r$ ,  $\Phi\Theta_0$  precedes  $\Phi'\Theta_1$ , and  $\exists i (\Theta_1[i] = \Theta_0[1])$  then  $\Phi\Theta_0 \preceq \Phi'\Theta_1$ . So the maximal word length in a bad sequence with respect to  $\preceq$  is bounded by

$$|\Gamma_0| + \sum (|R_r^{[i]}| - 1)$$

where  $\Gamma_0$  is the initial word and  $\sum (|R_r^{[i]}| - 1)$  runs over the set of different right-hand sides of all rules.

The upper bound is not exact due to the following two limitations. First, not every letter can be rewritten to the chosen right-hand side, i.e. the letter  $f$  cannot be rewritten to  $h$  in a one step. Second, some rules can accidentally share some letters in their right-hand sides. I.e. the letter  $g$  in the right-hand side of the rule  $f \rightarrow gf$  and the letter  $g$  in the right-hand side of the rule  $h \rightarrow g$  have different nature and the coincidence of the two letters is occasional. In the next section we show how to partly avoid this difficulty.

## 4 Annotated Prefix Grammars

If in the rules  $h \rightarrow g$  and  $f \rightarrow gf$  we write down the corresponding letters as e.g.  $g^{[f]}$  and  $g^{[h]}$  and say that  $g^{[f]} \not\approx g^{[h]}$  then the prefix grammar will generate computations with less number of occasional Turchin's pairs.

Let us give more formal definition of this sort of prefix grammars.

**Definition 6.** A prefix grammar  $\mathbf{G} = \langle \mathcal{Y}, \mathbf{R}, \Gamma_0 \rangle$ ,  $\mathbf{R} \subset \mathcal{Y}^+ \times \mathcal{Y}^*$  is called **annotated**<sup>3</sup> if

<sup>2</sup> In [9] it is also specified that  $|\Phi| > 0$ . If a computation is yielded by a grammar only with non-empty left-hand sides of rules then this limitation is unnecessary. Otherwise the condition  $|\Phi| > 0$  becomes essential to make the upper bound  $C'_{Max}$  constructed with a help of Lemma 1 exact.

<sup>3</sup> This grammar property 2 plays a role only in the construction of a longest bad sequences. So in most propositions grammars with only the properties 1 and 3 are also considered as annotated.

1. For every two rules  $R : R_l \rightarrow R_r, R' : R'_l \rightarrow R'_r$ , if  $\exists i, j, (R_r[i] \approx R'_r[j])$ , then  $R_r \approx R'_r$ ;
2. If  $R_l \rightarrow R_r \in \mathbf{R}$  and there is a rule  $R'_l \rightarrow R'_r$  in  $\mathbf{R}$  then  $R'_l \rightarrow R_r \in \mathbf{R}$ .
3. The initial word contains only unique letters:  $\forall i, j, k, (R_r^{[k]}[i] \neq \Gamma_0[j])$ .

Consider the following algorithm that transforms a prefix grammar  $\mathbf{G}$  to an annotated  $\mathbf{G}'$ .

1. Let  $a = R_r^{[n]}[i]$ ,  $a \in \mathcal{T}$ ,  $n$  be an unique number of the rule with the right-hand side  $R_r^{[n]}$ .  $a$  corresponds to the pair  $\langle a, 2^n * 3^{i-1} \rangle$ . We set  $n = 0$  for the initial word  $\Gamma_0$  and denote the corresponding tuple of the pairs  $\langle \Gamma_0[1], 1 \rangle \langle \Gamma_0[2], 3 \rangle \dots \langle \Gamma_0[|\Gamma_0|], 3^{|\Gamma_0|} \rangle$  as  $\Gamma'_0$ .
2. A rewrite rule  $R' : R'_l \rightarrow \Phi$  of the grammar  $\mathbf{G}'$  corresponds the the equivalence class up to left-hand sides of rules  $\langle a_i, n_i \rangle \rightarrow \Phi$ , where  $\Phi$  is a right-hand side of a rule from  $\mathbf{G}$  after the first step, and  $\langle a_i, n_i \rangle$  is arbitrary.

If the initial grammar  $\mathbf{G}$  yields a bad sequence then the computation by  $\mathbf{G}'$  that is derived from  $\Gamma'_0$  by application of the rules from the equivalence classes that correspond to the right-hand sides of the rules that are applied in the computation by  $\mathbf{G}$  is also a bad sequence.

We do not differ rewrite rules with the different left-hand sides in annotated grammars and write them as  $x \rightarrow R_r$  where  $x$  denotes an arbitrary pair sequence of a bounded length.

**Example 3** Let us transform the prefix grammar  $\mathbf{G}_{\text{LOG}}$  from Example 2 into an annotated.

$$\begin{array}{l}
 \mathbf{G}'_{\text{LOG}}: \\
 \Gamma_0 = \langle f, 1 \rangle \qquad R^{[2]} : x \rightarrow \langle g, 4 \rangle \\
 R^{[1]} : x \rightarrow \langle g, 2 \rangle \langle f, 6 \rangle \qquad R^{[3]} : x \rightarrow \langle h, 8 \rangle \\
 R^{[4]} : x \rightarrow \Lambda
 \end{array}$$

The computation by  $\mathbf{G}_{\text{LOG}}$  that corresponds to the computation from Example 2 now begins as follows:

$$\begin{array}{ccc}
 \Gamma_0 : \langle \mathbf{f}, \mathbf{1} \rangle_{(0)} & & \Gamma_2 : \langle \mathbf{h}, \mathbf{8} \rangle_{(3)} \langle \mathbf{f}, \mathbf{6} \rangle_{(1)} \\
 R^{[1]} \downarrow & \nearrow R^{[3]} & R^{[2]} \downarrow \\
 \Gamma_1 : \langle \mathbf{g}, \mathbf{2} \rangle_{(2)} \langle \mathbf{f}, \mathbf{6} \rangle_{(1)} & & \Gamma_3 : \langle \mathbf{g}, \mathbf{4} \rangle_{(4)} \langle \mathbf{f}, \mathbf{6} \rangle_{(1)}
 \end{array}$$

Note that now  $\Gamma_1 \not\leq \Gamma_3$ .

A useful feature of annotated grammars is their ability to generate longest bad sequences. There are no intersections in the right-hand sides of rewrite rules and thus  $\approx$  discerns prefixes that are yielded by distinct rule applications.<sup>4</sup>

<sup>4</sup> This can be very useful if there is a rule  $R$  with the left-hand side  $R_l$  embedded in the right-hand side  $R_r$ . Note that if  $R_l \leq R_r$  then the subsequence termination criterion is always activated after an application of the rule  $R_l \rightarrow R_r$ . This problem was pointed in [12].

Now we can find the upper bound of a bad sequence length in a computation yielded by a prefix grammar. The proof uses the following lemma.

**Lemma 1** *Every computation by an annotated prefix grammar ends either by  $\Lambda$  or by a Turchin pair  $\Phi\Theta_0, \Phi'\Psi\Theta_0$  such that there exists a rule  $R_l \rightarrow R_r$ , for which  $\Phi \approx R_r, \Phi' \approx R_r$ , and  $R_r \neq \Lambda$ .*

For the proof see Appendix. Note that the proof is for not only alphabetic prefix grammars but for prefix grammars that allow rules of the form  $\Phi \rightarrow \Psi$ . With the help of Lemma 1 we proved that the exact upper bound of a bad sequence length for an annotated prefix grammar is

$$C'_{Max} = |\Gamma_0| * (1 + |R_r^{[0]}| * (1 + |R_r^{[1]}| * (\dots * (1 + |R_r^{[N]}|) \dots)))$$

where rules in the sequence  $R^{[0]}, R^{[1]}, \dots, R^{[N]}$  are placed by a non-increasing order with respect to the length of their right-hand sides  $|R_r^{[i]}|$  (the proof of this fact is by induction; for details see [11]).

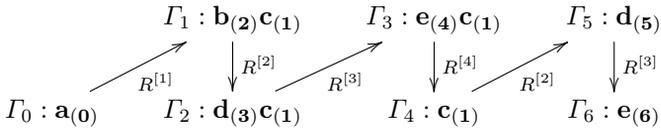
Note that  $N$  in the formula  $|\Gamma_0| * (1 + |R_r^{[0]}| * (1 + |R_r^{[1]}| * (\dots * (1 + |R_r^{[N]}|) \dots)))$  denotes not the cardinality of the set of rewrite rules but the cardinality of the set of the right-hand sides of rewrite rules. Thus when we do the annotation there is no exponential growth of the upper bound.

**Example 4** *Let us estimate the length of a longest bad sequence yielded by the grammar  $\mathbf{G}'_{LOG}$  (Example 3). The length of the initial word is 1,  $|R_r^{[1]}|$  has the length 2, and two rules have the right-hand sides of the length 1. The corresponding bad sequence length is 7.*

*Now let us build such bad sequence explicitly. For the sake of readability different pairs of the form  $\langle \text{letter}, \text{number} \rangle$  are denoted by different letters (thus  $\langle f, 1 \rangle = a, \langle g, 2 \rangle = c, \langle f, 6 \rangle = c, \langle g, 4 \rangle = d, \text{ and } \langle h, 8 \rangle = e$ ).*

$$\begin{array}{l} \mathbf{G}'_{LOG}: \\ \Gamma_0 = a \qquad R^{[2]} : x \rightarrow d \\ R^{[1]} : x \rightarrow bc \qquad R^{[3]} : x \rightarrow e \\ R^{[4]} : x \rightarrow \Lambda \end{array}$$

*One of the maximal bad sequences is:*



*Note that the segment  $\Gamma_5 - \Gamma_6$  cannot be generated by the initial grammar  $\mathbf{G}_{LOG}$ .*

If we aim to find embeddings not only in traces generated by single prefix grammars but also in direct products of the traces then usage of  $\preceq$  causes some

questions. Namely we must know whether well-binarity is preserved on intersections of the Turchin relation with some wqo. The problem is that the Turchin relation is not well binary on arbitrary computations' subsequences — we only can prove that it is well binary on the whole computations.

**Example 5** Consider the following computation yielded by a prefix grammar.

$$\begin{array}{ll}
 \Gamma_0 : \mathbf{a}_{(2)}\mathbf{b}_{(1)}\mathbf{c}_{(0)} & \Gamma_7 : \mathbf{b}_{(5)}\mathbf{c}_{(3)} \\
 \Gamma_1 : \mathbf{b}_{(1)}\mathbf{c}_{(0)} & \Gamma_8 : \mathbf{c}_{(3)} \\
 \Gamma_2 : \mathbf{c}_{(0)} & \Gamma_9 : \mathbf{b}_{(10)}\mathbf{c}_{(9)} \\
 \Gamma_3 : \mathbf{b}_{(4)}\mathbf{c}_{(3)} & \Gamma_{10} : \mathbf{a}_{(12)}\mathbf{b}_{(11)}\mathbf{c}_{(9)} \\
 \Gamma_4 : \mathbf{a}_{(6)}\mathbf{b}_{(5)}\mathbf{c}_{(3)} & \Gamma_{11} : \mathbf{a}_{(14)}\mathbf{a}_{(13)}\mathbf{b}_{(11)}\mathbf{c}_{(9)} \\
 \Gamma_5 : \mathbf{a}_{(8)}\mathbf{a}_{(7)}\mathbf{b}_{(5)}\mathbf{c}_{(3)} & \Gamma_{12} : \mathbf{a}_{(16)}\mathbf{a}_{(15)}\mathbf{a}_{(13)}\mathbf{b}_{(11)}\mathbf{c}_{(9)} \\
 \Gamma_6 : \mathbf{a}_{(7)}\mathbf{b}_{(5)}\mathbf{c}_{(3)} & \dots \dots
 \end{array}$$

No two elements of the sequence  $\Gamma_0, \Gamma_5, \Gamma_{12}, \Gamma_{21}, \dots$  form a Turchin pair.

The following lemma verifies well-binarity of the intersections (the proof is in Appendix).

**Lemma 2**  $\preceq$  contains a wqo  $T$  that is well binary on all computations yielded by an annotated prefix grammar.

$T$  contains Turchin pairs of a special sort. Namely it contains  $\langle \Gamma, \Delta \rangle$  such that  $\Gamma = \Phi\Theta_0, \Delta = \Phi'\Psi\Theta_0$ , there exists a rule  $R : R_l \rightarrow R_r$  with  $|R_r| > 0, R_r \approx \Phi, R_r \approx \Phi'$ , and  $\Psi[1]$  is never modified in the further computation. So the Turchin relation may not be checked after erasures with no loss of well-binarity.

What is more, existence of  $T$  guarantees that the Turchin relation can be intersected with an arbitrary wqo without loss of well-binarity. On the other hand, the idea of intersecting two Turchin relations looks appealing but implies a possible existence of infinite bad sequences with respect to the intersection.

**Definition 7.** Let us say that  $\Gamma$  is embedded in  $\Delta$  with not more than with a single gap if there exist words  $\Phi, \Psi, \Theta$  (maybe empty) such that  $\Gamma = \Phi\Theta, \Delta = \Phi\Psi\Theta$ .

Let us say that  $\Gamma$  is embedded in  $\Delta$  with not more than with  $n + 1$  gaps if there exist (maybe empty)  $\Phi, \Psi, \Theta_1, \Theta_2$  such that  $\Gamma = \Phi\Theta_1, \Delta = \Phi\Psi\Theta_2$  and  $\Theta_1$  is embedded in  $\Theta_2$  with not more than with  $n$  gaps.

Let us give a simple example. **abac** is embedded in **abrac** with not more than a single gap and in **abracadabra** — with not more than two gaps (**abac** is divided into only two parts **ab** and **ac**, but the end of a word is also considered as its part. The end of the word **abracadabra** is not at the same position as the end of **ac**, so the gap between **ac** and the end of the word is also taken into account).

**Lemma 3** If a relation  $R$  of word embedding allows only finite number of gaps then it is not well binary on sequences that are yielded by a direct product of prefix grammars  $G_1 \times G_2$ , even when the grammars are deterministic.

For the proof see Appendix. So the Turchin relation can be intersected with any relation that is well binary on the whole  $\{\mathcal{Y}^*\}$ , but not with the other Turchin relation.

### 5 Turchin's Relation and Subsequence Relation

The Turchin theorem not only guarantees existence of a Turchin pair for every infinite computation but also gives the exponential upper bound of a bad sequence length. In the case of computations yielded by annotated prefix grammars the upper bound of a bad sequence with respect to the subsequence relation coincides with the upper bound of a bad sequence  $C'_{Max} = |\Gamma_0| * (1 + |R_r^{[0]}| * (1 + |R_r^{[1]}| * (\dots * (1 + |R_r^{[N]}|) \dots)))$  for the Turchin relation. What is more, in a computation yielded by an annotated prefix grammar the lengths of bad sequences with respect to these two relations always coincide.

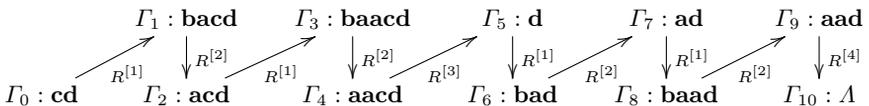
**Lemma 4** *The first pair in a computation yielded by an annotated prefix grammar that satisfy the subsequence relation is a Turchin pair.*

For the proof see Appendix.

Lemma 4 may have some practical meaning for systems of program transformation that use the homeomorphic embedding as a branch termination criterion.

**Example 6** *The left-hand side of the definition  $f(S(x))=S(f(g(S(x))))$  is embedded in the right-hand side in the sense of the subsequence relation. So an unfolding of the call  $f(S(Z))$  yields termination if the homeomorphic embedding is used as a termination criterion. If we use the intersection of this relation with the Turchin relation, the call does not cause the early termination<sup>5</sup>. The other way to prevent this too early termination is to use the annotated subsequence relation: it is only enough to annotate function calls to avoid unwanted embeddings with the effect similar to the usage of the Turchin relation intersected with  $\triangleleft$ .*

Then the question emerges if the annotated subsequence relation can prevent too early terminations for every prefix-grammar-generated computation. Namely, whether the annotated subsequence relation allows unfolding to find a trace ending by  $\Lambda$  if  $\Lambda$  is in the language of the prefix grammar. The answer is yes for alphabetic prefix grammars [10] and is negative in the general case when  $\mathbf{R} \subset \mathcal{Y}^* \times \mathcal{Y}^*$ . To illustrate the last claim, consider the grammar  $\mathbf{G}_\Lambda$  of Example 1 if  $\Gamma_0 = cd$ .  $\Lambda$  belongs to the language of the grammar since



<sup>5</sup> A termination criterion of this type is used in the supercompiler SCP4 [9].

The grammar belongs to the class of annotated grammars. But there are several pairs with respect to  $\trianglelefteq$  (and  $\preceq$ ) on the trace leading to  $\Lambda$ : e. g.  $\Gamma_1 \preceq \Gamma_3$ . To solve the empty word problem for languages generated by non-alphabetic prefix grammars using the subsequence relation as a termination criterion we need to do some more annotation, which is proved in [10].

Recall that in the case of pairs over  $\trianglelefteq$  the upper bound on arbitrary sequences with restricted word length growth is multiple recursive [15]. We show that the upper bound is exact even if the sequences are built by a direct product of two prefix grammars.

**Example 7** Consider the following rewrite grammar (it not necessarily rewrites only prefixes; the grammar is similar to the one described in [15]).

$$\begin{array}{lll} R^{[1]} : su \rightarrow ss & R^{[2]} : tu \rightarrow tt & R^{[3]} : ts \rightarrow tt \\ R^{[4]} : wu \rightarrow ws & R^{[5]} : tws \rightarrow utw & R^{[6]} : tw \rightarrow ws \\ R^{[7]} : sws \rightarrow wt & R^{[8]} : sw \rightarrow wsss & \end{array}$$

If the rules are applied to the initial word  $sss\dots swww\dots w$  then the trace of the length  $O(B(m, n))$  with no pairs with respect to  $\trianglelefteq$  is generated.

Now we build a system of two prefix grammars that models the example of H. Touzet ( $x$  denotes an arbitrary letter).

1. The first letter of a word rewritten by the first prefix grammar  $\mathbf{G}_1$  represents a current state of the Turing machine.
2. The last letter of a word rewritten by the second prefix grammar  $\mathbf{G}_2$  represents the end of data [EOW] and is always rewritten into itself.
3. The word rewritten by  $\mathbf{G}_1$  represents the initial fragment of data which is before the counter of Turing machine. The word rewritten by  $\mathbf{G}_2$  represents the final fragment of data which is behind the counter of Turing machine.
4. There is an auxiliary set of rules moving the counter to the beginning of the data  $\langle State_0x, y \rangle \rightarrow \langle State_0, xy \rangle$ .
5. There is a rule that starts the rewrite process  $\langle State_0, y \rangle \rightarrow \langle State_1^F, y \rangle$ .
6.  $R^{[i]} : R_i^{[i]} \rightarrow R_r^{[i]}$  are modeled by  $\langle State_i^F x, [EOW] \rangle \rightarrow \langle State_i^B x, [EOW] \rangle$  (if  $i \neq 8$ ), a rule  $\langle State_i^F R_i^{[i]}, x \rangle \rightarrow \langle State_0\Lambda, R_r^{[i]}x \rangle$  and a set of rewrite rules  $\langle State_i^F x, y \rangle \rightarrow \langle State_i^F xy, \Lambda \rangle$  where  $x$  does not coincide with  $R_i^{[i]}$ .
7. The set of rules  $\langle State_i^B x, y \rangle \rightarrow \langle State_i^B, xy \rangle$  is similar to the one for  $State_0$  but the last rule now looks like  $\langle State_i^B, y \rangle \rightarrow \langle State_{i+1}^F, y \rangle$  instead of  $\langle State_0, y \rangle \rightarrow \langle State_1^F, y \rangle$ .

If there are two pairs  $\langle a_1\Phi, \Psi \rangle, \langle a_2\Phi', \Psi' \rangle$  such that  $a_1\Phi \trianglelefteq a_2\Phi'$  and  $\Psi \trianglelefteq \Psi'$  then  $a_1 = a_2$  and  $\Phi\Psi \trianglelefteq \Phi'\Psi'$ . There can be no such pairs if  $a_2$  is not changed on the trace fragment from  $\langle a_1\Phi, \Psi \rangle$  to  $\langle a_2\Phi', \Psi' \rangle$  because then  $|\Psi'| < |\Psi|$ . If  $a_2$  is changed on the trace fragment from  $\langle a_1\Phi, \Psi \rangle$  to  $\langle a_2\Phi', \Psi' \rangle$  then one of the rules  $\langle State_i^F R_i^{[i]}, x \rangle \rightarrow \langle State_0\Lambda, R_r^{[i]}x \rangle$  is applied on the fragment and thus  $\Phi\Psi \not\trianglelefteq \Phi'\Psi'$ . Therefore the bad sequence length on the trace generated by  $\mathbf{G}_1 \times \mathbf{G}_2$  with respect to the subsequence relation must be also estimated by  $O(B(m, n))$ .

Now we can see that transition from only stack transformations to stack-plus-data transformations even in the unary case lifts the computational model from the finite automata up to full power of Turing machines. Thus it becomes interesting to investigate how the popular wqos work on intermediate prefix grammar constructions (as  $-2$  or  $-1$ -class prefix grammars [5]), which are widely used in term rewriting theory.

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## Appendix

*Proof (Lemma 1 on the first Turchin pair).*

Let us consider a pair  $\Phi_1\Theta_0, \Phi_2\Psi\Theta_0$  such that  $\Phi_1\Theta_0 \preceq \Phi_2\Psi\Theta_0$ ,  $(\Phi_1 \approx \Phi_2)$ , and the trace segment ending with  $\Phi_2\Psi\Theta_0$  is a bad sequence. According to the properties of annotated grammars,  $\Phi_1[1]$  and  $\Phi_2[1]$  must be generated by different applications of the same rule  $R : x \rightarrow R_r$  with  $|R_r| > 0$ , and if  $\Phi_1[1] \approx R_r[i]$  then necessarily  $\Phi_2[1] \approx R_r[i]$ . Let us denote the prefix  $R_r[1]R_r[2]\dots R_r[i-1]$  as  $R_{(z)}^{(i-1)}$  ( $z$  is the time index of  $R_r[i-1]$ ). Now turn back to the two applications of  $R$ . The result of the former must be of the form  $R_{(k_1)}^{(i-1)}\Phi_1\Theta_0$ , the result of the latter is of the form  $R_{(k_2)}^{(i-1)}\Phi_2\Psi\Theta_0$ . They form a Turchin pair and therefore coincide with  $\Phi_1\Theta_0$  and  $\Phi_2\Psi\Theta_0$ .

So  $\Phi_1 = R_{r_1}\Phi'_1, \Phi_2 = R_{r_2}\Phi'_2$  ( $\Phi'_1 \approx \Phi'_2$ , and  $R_{r_1}$  and  $R_{r_2}$  coincide up to the time indices with some right-hand side of a rewrite rule). Let  $\Phi'_1$  be non-empty. Then  $\exists R', j(\Phi'_1[1] \approx R'_r[j] \ \& \ \Phi'_2[1] \approx R'_r[j])$ , and  $\Phi'_1[1] \neq \Phi'_2[1]$ . The prefix  $R'_r[1]R'_r[2]\dots R'_r[j-1]$  is denoted as  $R_{(z)}^{(j-1)}$ . Now turn back to the  $R'$  applications that generate  $\Phi'_1[1]$  and  $\Phi'_2[1]$ . They look as  $R_{(l_1)}^{(j-1)}\Phi'_1\Theta_0$  and  $R_{(l_2)}^{(j-1)}\Phi'_2\Psi\Theta_0$  and form a Turchin pair. This contradicts the choice of  $\Phi_1\Theta_0$  and  $\Phi_2\Psi\Theta_0$ .

Hence  $\Phi_1\Theta_0 = R_{r_1}\Theta_0$  and  $\Phi_2\Psi\Theta_0 = R_{r_1}\Psi\Theta_0$ .

*Proof (Existence of a wqo subset of Turchin’s relation).*

Let  $(\mathcal{T}, \mathbf{R}, \Gamma_0)$  be an annotated prefix grammar  $\mathbf{G}$ . Consider all traces  $\{\Phi_i\}_{i=1}^\infty$  generated by  $\mathbf{G}$  such that  $\exists N \forall i \exists j (i < j \ \& \ |\Phi_j| \leq N)$ .

For every trace  $J$  from this set choose the least  $N$  that satisfies this property. Due to finiteness of the set  $\mathbf{R}$  words generated by the rules from  $\mathbf{R}$  can contain finite set of letters. Therefore some word  $\Psi$  of the length  $N$  must repeat itself (with respect to  $\approx$ ) infinitely in  $J$ . The first letter of  $\Psi$  is generated by a single rule  $R$  with a non-empty right-hand side. Every two results of these applications of  $R$  look as  $\Delta\Psi$  and  $\Delta'\Psi'$  where  $\Psi \approx \Psi'$  and  $\Delta \approx \Delta'$  for they are same prefixes of the same right-hand side  $R_r$  that end at  $\Psi[1]$  so  $\Delta\Psi$  and  $\Delta'\Psi'$  form a Turchin pair.

All other traces  $\{\Phi_i\}_{i=1}^\infty$  have an infinite growth of the minimal word length:  $\forall N \exists i_N \forall j (j > i_N \Rightarrow |\Phi_j| > N)$ . For every such trace and every  $N$  choose a minimal  $i_N$  such that all successors of  $\Phi_{i_N}$  never have the length less than  $N$ :  $\forall j (j < i_N \Rightarrow \exists k (k \geq j \ \& \ |\Phi_k| < N))$ . So  $|\Phi_{i_N-1}| < N, |\Phi_{i_N}| \geq N$ , and  $\Phi_{i_N}$  is generated from its predecessor by some  $R$  with a non-empty right-hand side,  $|R_r| \geq 2$ :  $\Phi_{i_N} = R_{r(l)}\Phi_{i_N-1}^-$  where  $\Phi_{i_N-1}^-$  is a suffix of  $\Phi_{i_N-1}$ .  $\Phi_{i_N-1}^-$  stays constant because  $|\Phi_{i_N-1}| < N$ . All the elements of  $\{\Phi_{i_N}\}_{N=1}^\infty$  begin with a

non-empty right-hand side of a some rewrite rule, therefore exists an infinite subsequence  $\{\Phi_{i_K}\}_{K=1}^\infty$  of  $\{\Phi_{i_N}\}_{N=1}^\infty$  such that all elements of  $\{\Phi_{i_K}\}_{K=1}^\infty$  begin with the right-hand side of a same rule. Every two elements of  $\{\Phi_{i_K}\}_{K=1}^\infty$  form a Turchin pair.

The set  $T$  of pairs of these two sorts is a wqo on traces generated by an annotated prefix grammar.

**Example 8 (Non-well-binarity of  $N$ -gaps relation)** *Proof.* Let us consider a class of grammars  $\mathbf{G}^{[n]}$  on pairs of words in the alphabet  $\{a_1, \dots, a_n, A_1, \dots, A_n, e, E\} \times \{a_1, \dots, a_n, A_1, \dots, A_n, e, E\}$ , with the initial word  $\langle e, A_1 A_2 \dots A_n E \rangle$  and the following rewrite rules:

- $R^{[00]} : \langle e, a_1 \rangle \rightarrow \langle E, a_1 a_1 \rangle$
- $R^{[01]} : \langle E, A_1 \rangle \rightarrow \langle e, A_1 A_1 \rangle$
- $R^{[02]} : \langle e, A_1 \rangle \rightarrow \langle a_1 e, \Lambda \rangle$
- $R^{[03]} : \langle E, a_1 \rangle \rightarrow \langle A_1 E, \Lambda \rangle$
- ...
- $R^{[i0]} : \langle a_i, a_i \rangle \rightarrow \langle \Lambda, a_i a_i \rangle$
- $R^{[i1]} : \langle A_i, A_i \rangle \rightarrow \langle \Lambda, A_i A_i \rangle$
- $R^{[i2]} : \langle a_i, A_i \rangle \rightarrow \langle a_i a_i, \Lambda \rangle$
- $R^{[i3]} : \langle A_i, a_i \rangle \rightarrow \langle A_i A_i, \Lambda \rangle$
- $R^{[i4]} : \langle a_i, a_{i+1} \rangle \rightarrow \langle \Lambda, a_i a_{i+1} a_{i+1} \rangle$
- $R^{[i5]} : \langle A_i, A_{i+1} \rangle \rightarrow \langle \Lambda, A_i A_{i+1} A_{i+1} \rangle$
- $R^{[i6]} : \langle a_i, A_{i+1} \rangle \rightarrow \langle a_{i+1} a_i a_i, \Lambda \rangle$
- $R^{[i7]} : \langle A_i, a_{i+1} \rangle \rightarrow \langle A_{i+1} A_i A_i, \Lambda \rangle$
- ...
- $R^{[n0]} : \langle a_n, e \rangle \rightarrow \langle \Lambda, a_n e \rangle$
- $R^{[n1]} : \langle A_n, E \rangle \rightarrow \langle \Lambda, A_n E \rangle$
- $R^{[n2]} : \langle a_n, E \rangle \rightarrow \langle a_n a_n, e \rangle$
- $R^{[n3]} : \langle A_n, e \rangle \rightarrow \langle A_n A_n, E \rangle$

For every  $N$  there exists some  $n$  such that  $\mathbf{G}^{[n]}$  yields a trace with no pair  $\langle \Phi_1, \Psi_1 \rangle, \langle \Phi_2, \Psi_2 \rangle$  such that  $\Phi_1$  is embedded in  $\Phi_2$  and  $\Psi_1$  is embedded in  $\Psi_2$  with not more than  $N$  gaps.

*Proof (The proof of Lemma 4).* Consider  $\Phi_1$  and  $\Phi_2$  such that  $\Phi_1 \preceq \Phi_2$ , there are no pairs with respect to the subsequence relation in the trace before  $\Phi_2$ , and  $\Phi_1$  is embedded into  $\Phi_2$  with  $n + 1$  gaps (up to time indices). Let  $\Theta_0$  be their common suffix. Then  $\Phi_1 = A_1 A_2 \dots A_n \Theta_0$  and  $\Phi_2 = B_1 A'_1 B_2 A'_2 \dots B_n A'_n B_{n+1} \Theta_0$ , where  $A_i \approx A'_i$  for all  $i$  from 1 to  $n$ . Consider the words where letters  $A_n[1]$  and  $A'_n[1]$  are generated. The grammar features guarantee that both letters are generated by a same rule  $R : x \rightarrow R_r$ , so the words look as  $\Delta A_n \Theta_0$  and  $\Delta' A'_n B_{n+1} \Theta_0$ .  $\Delta \approx \Delta'$  for they are same prefixes of the same right-hand side. This implies that  $\Delta A_n \Theta_0 \preceq \Delta' A'_n B_{n+1} \Theta_0$ . In the computation  $\Phi_1$  and  $\Phi_2$  are the first pair with respect to the subsequence relation, consequently  $i = 1 = n$ , and  $\Phi_1 = A_1 \Theta_0$ , and  $\Phi_2 = A'_1 B_{n+1} \Theta_0$ , so  $\Phi_1 \preceq \Phi_2$ .